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2008 J. Phys. A: Math. Theor. 41 475001

(http://iopscience.iop.org/1751-8121/41/47/475001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 475001 (9pp) doi:10.1088/1751-8113/41/47/475001

Localization for one-dimensional random potentials with large local fluctuations

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Received 4 July 2008, in final form 9 September 2008
Published 20 October 2008
Online at stacks.iop.org/JPhysA/41/475001

Abstract

We study the localization of wavefunctions for one-dimensional Schrödinger Hamiltonians with random potentials $V(x)$ with short-range correlations and large local fluctuations such that $\int \mathrm{d}x \langle V(x)V(0)\rangle = \infty$. A random supersymmetric Hamiltonian is also considered. Depending on how large the fluctuations of $V(x)$ are, we find either new energy dependences of the localization length, $\ell_{\mathrm{loc}} \propto E/\ln E$, $\ell_{\mathrm{loc}} \propto E^{\mu/2}$ with $0 < \mu < 2$ or $\ell_{\mathrm{loc}} \propto \ln^{\mu-1} E$ for $\mu > 1$, or superlocalization (decay of the wavefunctions faster than a simple exponential).

PACS numbers: 72.15.Rn, 73.20.Fz, 02.50.−r

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The phenomenon of Anderson localization [1] in one dimension has been widely studied since the pioneering work of Mott & Twose arguing that all states are localized in one dimension (1D) [2]. This statement was rigorously proven in [3, 4]. A general method to study the spectral and localization properties of 1D random Hamiltonian was proposed in [5, 6]: let us consider the one-dimensional Schrödinger Hamiltonian $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ where $V(x)$ is a random potential with short-range correlations. We first study the solution of the stationary Schrödinger equation $H\psi(x;E) = E\psi(x;E)$ satisfying $\psi(0;E) = 0$ and $\psi'(0;E) = 1$ (differentiation with respect to x is denoted by $'$). We define the Lyapunov exponent (the inverse localization length $\ell_{\mathrm{loc}} \equiv 1/\gamma$) as the increase rate [5, 7]

$$\gamma(E) \stackrel{\mathrm{def}}{=} \lim_{x\to\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left\langle \ln\sqrt{\psi(x;E)^2 + \frac{1}{E}\psi'(x;E)^2}\right\rangle. \tag{1}$$

Averaging $\langle\cdots\rangle$ is taken over realizations of the random potential. This definition becomes more clear if the wavefunction is parametrized in terms of an envelope and an oscillating part.

We substitute to the couple of functions (ψ, ψ') the variables (θ, ξ) according to

$$\psi(x; E) = \mathrm{e}^{\xi(x)} \sin\theta(x) \tag{2}$$

$$\psi'(x; E) = k\, \mathrm{e}^{\xi(x)} \cos\theta(x), \tag{3}$$

where $E = k^2$. We can rewrite the definition of the Lyapunov exponent as $\gamma(E) = \frac{\mathrm{d}}{\mathrm{d}x}\langle\xi(x)\rangle$. Therefore the Lyapunov exponent gives the rate of the exponential increase of the envelope of the wavefunction.

At high energy (compared to disorder), oscillations of the wavefunction occur on the typical scale k^{-1} and the Lyapunov exponent is given by [5, 6][1]

$$\gamma(E \to \infty) \simeq \frac{1}{8k^2} \int \mathrm{d}x \langle V(x)V(0)\rangle \cos 2kx. \tag{4}$$

The question of the present paper is to discuss the situation where the potential presents large fluctuations such that $\int \mathrm{d}x \langle V(x)V(0)\rangle = \infty$, which makes (4) inapplicable. The integral of the correlation function may diverge for different reasons. First, it may diverge due to long-range correlations. This corresponds to nonstationary potentials. The localization for self-affine potentials such that $\langle[V(x) - V(0)]^2\rangle \propto |x|^{2h}$, with the Hurst exponent $h > 0$, was studied in [9–11] (the case $h = 1/2$ corresponds to the Brownian case, partly studied in [12]). This will not be the question of interest in the present paper. Another reason for the divergence $\int \mathrm{d}x \langle V(x)V(0)\rangle = \infty$ is for a potential with *short-range* correlations and *large local* fluctuations. This is the case on which we will focus here.

For simplicity we consider a random potential $V(x)$ uncorrelated at different positions (vanishing correlation length). A model that realizes these conditions is the following random potential:

$$V(x) = \sum_n v_n \delta(x - x_n), \tag{5}$$

where the weights v_n are chosen to be independent and identical random variables distributed according to a distribution with power law tail

$$p_1(v) \propto \frac{1}{w}\left|\frac{w}{v}\right|^{1+\mu} \qquad \text{for} \quad v \to \pm\infty \tag{6}$$

with $\mu > 0$. Here w is a scale for the weights to make the argument of the tail dimensionless. The positions of impurities x_n are also chosen to be independent random variables uniformly distributed with a finite density ρ. When $\mu \leqslant 2$ the second moment diverges $\langle v_n^2\rangle = \infty$, as well as the correlation function of the potential (5) since $\langle V(x)V(x')\rangle = \rho\langle v_n^2\rangle\delta(x - x')$.

Some exact results have been obtained for a tight binding Hamiltonian with random on-site potential (the Anderson model) distributed according to a Cauchy law [13–15] (section 10.3 of [6] or [7]). Note also that fluctuations of the envelope of the wavefunction and conductance statistics for this discrete model were studied for power law disorder in the recent works [16, 17]. The Anderson model (AM) can be mapped [7] onto the problem we are interested in here for *fixed* impurity positions and with $\mu = 1$. Potentials for fixed and random impurity

[1] The solution $\psi(x; E)$ of the Cauchy problem exists for any value of the energy; it is used to construct the normalized wavefunctions $\varphi(x)$ of the stationary Schrödinger equation on a finite interval $[0, L]$ satisfying boundary conditions $\varphi(0) = \varphi(L) = 0$, what can only be achieved for a discrete set of energies (the Sturm–Liouville problem). From this scheme we expect that the normalized wavefunctions present the structure $\varphi(x) \sim \sin(kx + \theta_0)\,\mathrm{e}^{-|x-x_0|/\ell_{\mathrm{loc}}}$. Note however that this simple picture neglects the important fact that, in the exponential, $\xi(x)$ has large absolute fluctuations despite it presents negligible fluctuations relatively to its average when $\int \mathrm{d}x\langle V(x)V(0)\rangle < \infty$. These fluctuations play a very important role since they induce large fluctuations of the normalization of the wavefunction (see section 13.3 of [6]) [8].

positions share some features, however, when impurities of random weights form a lattice, $x_n = n/\rho$, the trace of the lattice remains for arbitrary large energies (band edges remain at $k_n = n\pi\rho$). This makes the definition of a high energy regime less convenient.

2. Ricatti variable

We follow the ideas introduced in [18] in order to study the spectrum, and apply them to the localization problem. Let us introduce the Ricatti variable $z(x) \stackrel{\text{def}}{=} \frac{\psi'(x)}{\psi(x)}$. From the Schrödinger equation we see that it obeys a Langevin equation $z' = -E - z^2 + V(x)$ for the initial condition $z(0) = \infty$. The distribution $T(z;x)$ of the Ricatti variable obeys the integro-differential equation

$$\frac{\partial}{\partial x}T(z;x) = \frac{\partial}{\partial z}[(E+z^2)T(z;x)] + \rho\int \mathrm{d}v\, p(v)[T(z-v;x) - T(z;x)]. \tag{7}$$

The first term on the right-hand side is the drift term coming from the force $-(E+z^2)$ and the second a jump term originating from the random potential (5). For x sufficiently large, the distribution reaches a limiting distribution $T(z)$ for a steady current [6, 7]. Current of the Ricatti variable through $\mathbb{R}$ gives the number of zeros of the wavefunction per unit length. This is also the integrated density of states (IDoS) per unit length $N(E)$, therefore

$$N(E) = (E+z^2)T(z) - \rho\int \mathrm{d}v\, p(v)\int_{z-v}^{z}\mathrm{d}z'\, T(z'). \tag{8}$$

Imposing normalization of the solution of this integral equation gives the IDoS. Knowing the limiting distribution $T(z)$, the Lyapunov exponent can be obtained from [6] $\gamma = \langle z\rangle$. Since $T(z\to\pm\infty) \simeq N(E)/z^2$, in order to deal with well-defined integral it is understood that the calculation of the Lyapunov exponent involves the antisymmetric part of the distribution: $\gamma = \int \mathrm{d}z\, z\frac{1}{2}[T(z) - T(-z)]$).

Let us study the high energy Lyapunov exponent. For that purpose we solve the integral equation (8) by perturbation starting from the solution for $V(x) = 0$. In the absence of the disorder ($p(v) = \delta(v)$) we have $T_0(z) = \frac{1}{\pi}\frac{k}{z^2+k^2}$. We expand the distribution $T(z) = T_0(z) + T_1(z) + \cdots$ in powers of the density ρ, as well as the IDoS. Then equation (8) is solved recursively order by order. We easily obtain $T_1(z)$ from which we deduce

$$\gamma(E = k^2 \to \infty) \simeq \frac{\rho}{\pi}\int \mathrm{d}z\,\frac{z}{z^2+k^2}\int \mathrm{d}v\, p(v)\left[\arctan\frac{z}{k} - \arctan\frac{z-v}{k}\right]. \tag{9}$$

This gives the general formula

$$\gamma(k^2) \simeq \frac{\rho}{2}\left\langle \ln\left[1+\left(\frac{v}{2k}\right)^2\right]\right\rangle_v, \tag{10}$$

where the averaging is now taken over the δ-peak weights v_n. This is the first term of a 'concentration expansion' that can be systematically developed [6] (equation (10) was derived in section 10.4 of this latter reference for non-random weights v_n. Additional averaging in equation (10) follows from the property of additivity of the variable $\xi(x)$).

3. New energy dependences

We first consider the high energy Lyapunov exponent, $\sqrt{E} = k \gg \rho, w$, when the weights are distributed according to (6). We write $p_1(v) = \frac{1}{w}f(v/w)$ where $f(y)$ is a dimensionless symmetric function such that $f(y\to\pm\infty) \simeq C|y|^{-1-\mu}$. We divide the integral $\gamma \simeq \rho \int_0^\infty \mathrm{d}y\, f(y) \ln\left[1+\left(\frac{w}{2k}y\right)^2\right]$ into three parts: $\gamma \sim \rho\left[\left(\frac{w}{2k}\right)^2 \int_0^1 \mathrm{d}y\, f(y) y^2 + \left(\frac{w}{2k}\right)^2 C \int_1^{2k/w} \mathrm{d}y\, y^{1-\mu} + 2C \int_{2k/w}^\infty \mathrm{d}y\, y^{-1-\mu} \ln\left(\frac{w}{2k}y\right)\right]$.

- For $\mu > 2$ the Lyapunov exponent is dominated by the smallest y ($\lesssim 1$). We obtain $\gamma \propto \rho\left(\frac{w}{k}\right)^2$ that corresponds to expand the logarithm of equation (10) for small v. This is the result of equation (4): $\gamma(k^2) \simeq \frac{1}{8k^2}\rho\langle v^2\rangle$.
- For $\mu = 2$, equation (4) cannot be used since $\langle v^2\rangle = \infty$. The integral giving the Lyapunov exponent is dominated by the intermediate scale $1 \lesssim y \lesssim k/w$:

$$\gamma(k^2) \propto \rho \left(\frac{w}{k}\right)^2 \ln\left(\frac{2k}{w}\right). \tag{11}$$

- For $0 < \mu < 2$, the Lyapunov exponent is dominated by the largest y ($\gtrsim k/w$). We obtain

$$\gamma(k^2) \propto \rho \left(\frac{w}{k}\right)^\mu. \tag{12}$$

The numerical dimensionless prefactors depend on the precise form of the distribution and not only on its tail.

The fluctuations of the random weights can be further increased by considering a distribution with tail:

$$p_2(v) \propto \frac{1}{|v| \ln^{1+\mu}\left|\frac{v}{w}\right|} \qquad \text{for} \quad v \to \pm\infty, \tag{13}$$

for $\mu > 0$. When $\mu > 1$ we find that the Lyapunov exponent decays logarithmically with energy

$$\gamma(k^2) \sim \frac{\rho}{\ln^{\mu-1}\left(\frac{k}{w}\right)}. \tag{14}$$

The case $0 < \mu \leqslant 1$ is discussed in the following section.

4. Superlocalization

On the other hand, for distribution (13) with $\mu \leqslant 1$, not only the second moment diverges $\langle v^2\rangle = \infty$, but the expression (10) shows that the Lyapunov exponent diverges as well: $\gamma = \infty$. This indicates that the logarithm of the envelope of the wavefunction, $\xi(x)$, presents different scaling properties with x. In order to analyze this, we remark that the variable $\xi(x)$ is constant between two impurities and makes a jump $\Delta\xi_n \stackrel{\text{def}}{=} \xi\left(x_n^+\right) - \xi\left(x_n^-\right) \sim \ln|v_n|$ across the impurity n (see below, the section on numerics). Therefore $\xi(x)$ behaves as the sum of $N \sim \rho x$ independent variables, each distributed according to a power law distribution $p(\Delta\xi) \propto \Delta\xi^{-1-\mu}$. Using well-known results (recalled in appendix A) we obtain

$$\xi(x) \sim (\rho x)^{1/\mu} \qquad \text{for} \quad 0 < \mu < 1 \tag{15}$$

$$\sim (\rho x)\ln(\rho x) \qquad \text{for} \quad \mu = 1. \tag{16}$$

The envelope of the wavefunction presents a decay faster than a simple exponential. This phenomenon is called *superlocalization* and has been recently studied for a discrete model in [19][2]. Characterization of the localization properties cannot be limited to the typical

[2] Note also that such superlocalization $\xi(x) \sim x^{1+h/2}$ occurs for self-affine random potentials characterized by long-range correlations $\langle[V(x) - V(0)]^2\rangle \propto |x|^{2h}$ with $h > 0$ [11].

behaviours (15), (16) since the variable $\xi(x)$ presents large fluctuations. Its distribution is characterized by a power law tail

$$\mathcal{P}(\xi; x) \propto 1/\xi^{1+\mu} \tag{17}$$

with the same exponent as that involved in the distribution of the weights.

4.1. Conductance

We give another interpretation of the previous result in terms of the conductance of a finite disordered interval of the length L. The dimensionless conductance is equal to the transmission probability (the Landauer formula), and presents the same exponential decay as the square of the wavefunction modulus. Therefore we can write $g \sim \mathrm{e}^{-2\xi(L)}$ (a more precise definition of the reflection coefficient within the phase formalism can be found in [5]).

Let us first recall some well-known results valid for a potential with finite local fluctuations. At high energy, when (4) holds, $\xi(x)$ behaves like a Brownian motion with drift[3] [5]: $\xi(x) \overset{\text{(law)}}{=} \gamma x + \sqrt{\gamma}W(x)$, where $W(x)$ is a Wiener process[4]. It follows that the distribution of the logarithm of the conductance is Gaussian $\Pi(\ln g) \simeq \frac{1}{\sqrt{8\pi\gamma L}} \exp -\frac{1}{8\gamma L}(\ln g + 2\gamma L)^2$ [21]. The typical value of the conductance is $g_{\text{typ}} \sim \mathrm{e}^{-6\gamma L}$ (while $(\ln g)_{\text{typ}} \sim -2\gamma L$) however fluctuations of the logarithm are associated with a much larger scale $g_{\text{fluct}} \sim \mathrm{e}^{-2\sqrt{\gamma L}}$.

Distribution of the conductance in the Anderson model with power law disorder has been studied in [17] where some power law distribution of the conductance was obtained for $g \to 0$.

In the superlocalization regime, the behaviour (15) is associated with a decay of the conductance $g \sim \mathrm{e}^{-L^{1/\mu}}$. The distribution (17) can be related to the conductance distribution:

$$\Pi(\ln g) \underset{g\to 0}{\sim} \frac{1}{|\ln g|^{\mu+1}} \tag{18}$$

for $0 < \mu \leqslant 1$.

5. Localization for supersymmetric Hamiltonian

We consider another class of random Hamiltonians with the so-called supersymmetric structure

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)^2 + \phi'(x) \qquad \text{with} \qquad \phi(x) = \sum_n \eta_n \delta(x - x_n), \tag{19}$$

where η_n are dimensionless uncorrelated weights, each distributed according to a distribution $p(\eta)$. This Hamiltonian is interesting since it presents rather different spectral and localization properties (in particular it leads to a delocalization transition as $E \to 0$). It is related to several other problems as well. For example it is the square of a Dirac Hamiltonian with a random mass $\phi(x)$, introduced in various contexts of condensed matter physics; the problem can also be related to classical diffusion in a random force field (see [6, 22, 23] for a review). We can follow the same strategy: the Ricatti variable $z = \frac{\psi'}{\psi} - \phi$ obeys the Langevin-type equation $z' = -E - z^2 - 2z\phi(x)$ with multiplicative noise. Limiting distribution of the Ricatti variable for a steady current $-N(E)$ obeys the integral equation

$$N(E) = (E + z^2)T(z) + \rho \int \mathrm{d}\eta\, p(\eta) \int_z^{z\mathrm{e}^{2\eta}} \mathrm{d}z'\, T(z'). \tag{20}$$

[3] The fact that drift and the Gaussian fluctuations involve the same parameter is referred to as '*single parameter scaling*' [20] (see also [16]); it holds only at high energy since it relies on the decoupling between a fast variable (the phase $\theta(x)$ introduced above) and the slow variable $\xi(x)$.

[4] a normalized free Brownian motion such that $\langle W(x)\rangle = 0$ and $\langle W(x)W(x')\rangle = \min(x, x')$.

The Lyapunov exponent is now given by $\gamma = \langle z\rangle + \langle\phi\rangle$. Following the same perturbative approach as before, we obtain

$$\gamma(E \to \infty) \simeq \rho\langle\ln\cosh\eta\rangle_\eta. \tag{21}$$

For the supersymmetric Hamiltonian, when the average exists, the Lyapunov exponent reaches a finite value at high energy (in contrast to the decrease of the Lyapunov exponent for the Schrödinger Hamiltonian): for $|\eta_n| \ll 1$ it takes the form $\gamma(E \to \infty) \simeq \frac{1}{2}\int \mathrm{d}x\, \langle\phi(x)\phi(0)\rangle$.

Let us consider a power law distribution of weights $p(\eta) \propto 1/|\eta|^{1+\mu}$. We see from (21) that we have $\gamma = \infty$ in this case for $\mu \leqslant 1$. The reason is similar to that discussed in the previous paragraph for the Schrödinger equation with weights distributed according to (13). Here the variable $\xi(x)$ jumps by $\Delta\xi_n \sim |\eta_n|$ across the impurity (see below). Therefore, for the supersymmetric case, a power law distribution of the weights leads to the superlocalization, equations (15)–(17).

6. Numerical calculations

We can easily study the evolution of the phase and envelope variables (2), (3) numerically. We denote by $\theta_n^\pm \stackrel{\text{def}}{=} \theta\left(x_n^\pm\right)$ and $\xi_n^\pm \stackrel{\text{def}}{=} \xi\left(x_n^\pm\right)$ the value of the phase and the envelope just before and right after the nth δ-peak. Between two impurities we have $\theta_{n+1}^- - \theta_n^+ = k\ell_n$ and $\xi_{n+1}^- - \xi_n^+ = 0$. The length $\ell_n = x_{n+1} - x_n$ denotes the distance between consecutive impurities. It is distributed according to a Poisson law $p(\ell) = \rho\,\mathrm{e}^{-\rho\ell}$. The evolution of the random variables across an impurity depends on the form of the random potential. We introduce the notation $\Delta\xi_n = \xi_n^+ - \xi_n^- = \xi_{n+1}^- - \xi_n^-$.

- For the *Schrödinger Hamiltonian* with potential (5) phase evolution is given by $\operatorname{cotg}\theta_n^+ - \operatorname{cotg}\theta_n^- = \frac{v_n}{k}$ and evolution of the envelope by $\Delta\xi_n = \ln\frac{\sin\theta_n^-}{\sin\theta_n^+} = \frac{1}{2}\ln\left[1 + \frac{v_n}{k}\sin 2\theta_n^- + \frac{v_n^2}{k^2}\sin^2\theta_n^-\right]$.
- For the *supersymmetric Hamiltonian* (19) we have $\tan\theta_n^+ = \mathrm{e}^{2\eta_n}\tan\theta_n^-$ and $\Delta\xi_n = \frac{1}{2}\ln\frac{\sin 2\theta_n^-}{\sin 2\theta_n^+} = \frac{1}{2}\ln[\mathrm{e}^{2\eta_n}\sin^2\theta_n^- + \mathrm{e}^{-2\eta_n}\cos^2\theta_n^-]$.

IDoS is given by $N(E) = \lim_{L\to\infty}\frac{\theta(L)}{L\pi}$ and the Lyapunov exponent by $\gamma(E) = \lim_{L\to\infty}\frac{\xi(L)}{L}$.

To be precise we consider a specific distribution with the power law tail :

$$p_1(v) = \frac{\mu\left|\frac{v}{w}\right|^{\mu-1}}{\pi w\left(1 + \left|\frac{v}{w}\right|^{2\mu}\right)}. \tag{22}$$

This choice has the advantage that it is very easy to simulate since the cumulative distribution is straightforwardly obtained. Using equation (10), we get the high energy Lyapunov exponent: $\gamma \simeq \frac{1}{\sin\left(\frac{\pi\mu}{2}\right)}\rho\left(\frac{w}{2k}\right)^\mu$ for $0 < \mu < 2$ and $\gamma \simeq \frac{2}{\pi}\rho\left(\frac{w}{2k}\right)^2\ln\left(\frac{2k}{w}\right)$ for $\mu = 2$ (both expressions are valid for $k \gg \rho, w$). The case $\mu = 1$ corresponds to a Cauchy law. The right-hand side of equation (10) can be computed easily in this case and we obtain the expression $\gamma \simeq \rho\ln\left(1+\frac{w}{2k}\right)$ valid in a broader range of energy (for $k \gg \rho$ but w arbitrary); at high energy we recover the known energy dependence $\gamma \simeq \rho\frac{w}{2k}$ (it can be recovered from discrete models [14, 7]). These expressions are compared to the numerical results on figure 1 and work perfectly well.

Next we analyze the superlocalization regime: we consider the supersymmetric Hamiltonian for weights η_n distributed according to a law similar to (22) for $0 < \mu < 1$. The distribution $\mathcal{P}(\xi; x)$ is plotted for different values of x on figure 2. In the inset the axes are rescaled in order to check that, according to (15), the distribution has the form

$$\mathcal{P}(\xi; x) \simeq \frac{1}{(\rho x)^{1/\mu}}\varpi\left(\frac{\xi}{(\rho x)^{1/\mu}}\right), \tag{23}$$

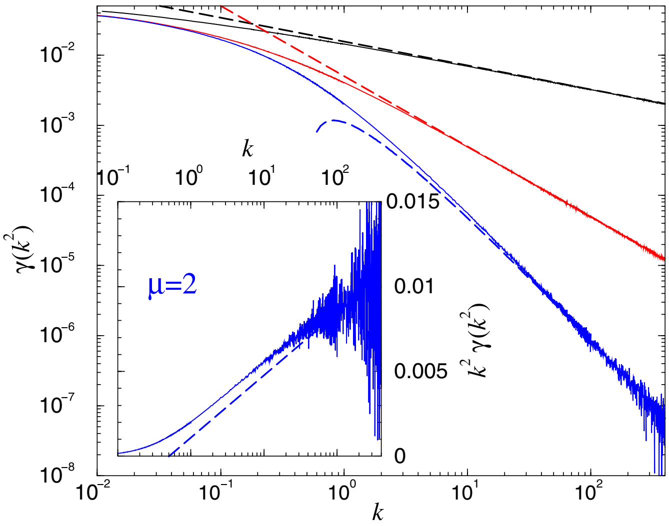

Figure 1. For the Schrödinger Hamiltonian: Lyapunov exponent as a function of the energy for weights v_n distributed according to (22). Numerical results (continuous lines) are compared with high energy expressions (dashed lines) derived in the text (no fit) for $\mu = 0.34$, 1 and 2. Other parameters are $w = 1$, $\rho = 0.01$ and number of impurities $N = 10^6$. Inset : $E\gamma(E)$ is plotted in semilog scale for $\mu = 2$ in order to check its logarithmic behaviour (dashed line corresponds to equation (11)).

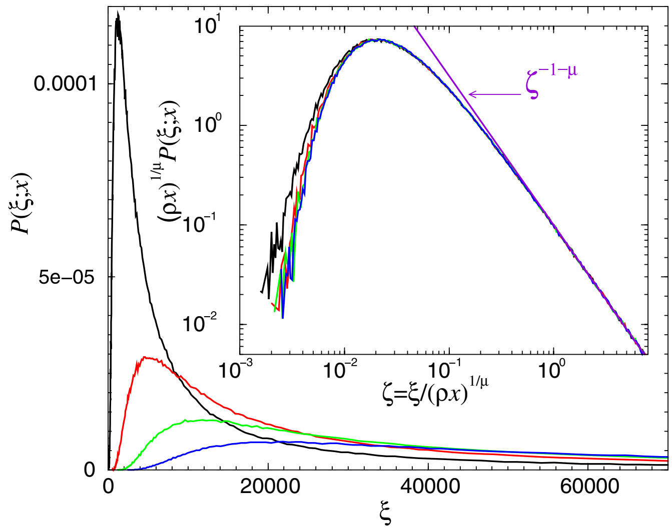

Figure 2. Superlocalization for the supersymmetric Hamiltonian. Distribution of the variable $\xi(x)$ for different values of $x = L/4$, $L/2$, $3L/4$ and L. Parameters are: $k = 10$, $\rho = 1$, $\Lambda = 0.1$ (typical scale for weights η_n) and $\mu = 0.5$. Number of impurities is $N = 10^6$ and $L = 1000$. In the inset, straight line corresponds to equation (17).

where $\varpi(\zeta)$ is a dimensionless function. After rescaling we see that the four curves corresponding to different values of x perfectly collapse onto each other, apart from small deviations corresponding to the smallest values of ξ and x. Finally we check that the tail of the

Table 1. Energy dependence of the localization length for the Schrödinger Hamiltonian with the random potential $V(x) = \sum_n v_n \delta(x - x_n)$ for different broad distributions of the weights v_n.

Potential distribution		Localization length
$\langle v_n^2 \rangle < \infty$		$\ell_{\mathrm{loc}} \propto E$ [6]
$p(v \to \pm\infty) \propto 1/\lvert v\rvert^{\mu+1}$	$\mu = 2$	$\ell_{\mathrm{loc}} \propto E/\ln E$
	$0 < \mu < 2$	$\ell_{\mathrm{loc}} \propto E^{\mu/2}$
$p(v \to \pm\infty) \propto \frac{1}{\lvert v\rvert} \ln^{-1-\mu} \left\lvert \frac{v}{w} \right\rvert$	$\mu > 1$	$\ell_{\mathrm{loc}} \propto \ln^{\mu-1} E$
Superlocalization	$\mu = 1$	$\xi(x) \sim x \ln x$
	$0 < \mu < 1$	$\xi(x) \sim x^{1/\mu}$

distribution is indeed a power law, equation (17): in the inset of figure 2 rescaled distributions are plotted on a log–log scale with $\varpi(\zeta) \propto \zeta^{-1-\mu}$. The agreement seems excellent.

7. Conclusion

We have analyzed the high energy localization length for random potentials with short-range correlations and large local fluctuations such that the well-known result (4), leading to $\ell_{\mathrm{loc}} \propto E$, is not valid. We have studied localization for potentials made of superposition of δ-peaks. Performing a concentration expansion, we have obtained two general high energy formulae for the Lyapunov exponent: $\gamma(E) \simeq \rho \left\langle \ln \left[1 + \frac{v^2}{4E}\right] \right\rangle_v$ [6] for the Schrödinger Hamiltonian and $\gamma(E) \simeq \rho \langle \ln \cosh \eta \rangle_\eta$ for the supersymmetric Hamiltonian. These formulae have been used to analyze the case of potential with large local fluctuations.

For the Schrödinger case, we have shown the relation between the distribution of the weights of the δ-peaks and the energy decay of the Lyapunov exponent (the inverse localization length ℓ_{loc}). Sufficiently large fluctuations of the weights, such that $\langle v_n^2 \rangle = \infty$, lead to a stronger localization effect characterized by an increase of ℓ_{loc} with energy slower than linear. These results are summarized in table 1.

The understanding of fluctuations of the variable $\xi(x)$ (i.e. of the localization length) plays a major role to analyze universal statistical properties of Wigner time delay [8]. It would be an interesting issue to study how the statistics of Wigner time delay are affected by the unconventional localization properties analyzed here.

Acknowledgments

We thank Alain Comtet and Satya Majumdar for interesting remarks and Jean-Marc Luck for bringing to our attention to [11, 19].

Appendix

We recall well-known results on the distribution of the sum of independent and identically distributed (*i.i.d*) random variables. Let us consider N *i.i.d.* positive variables y_n and their sum $Y_N = \sum_{n=1}^N y_n$. If $\langle y_n^2 \rangle < \infty$ the statistical properties of Y_N are given by the central limit theorem for $N \to \infty$: Gaussian distribution centred on $\langle Y_N \rangle = N\langle y \rangle$ for a variance $\langle Y_N^2 \rangle_c = \langle Y_N^2 \rangle - \langle Y_N \rangle^2 = N\langle y^2 \rangle_c$. If the distribution of the y_n's presents a power law tail $p(y) \propto 1/y^{\mu+1}$ with $0 < \mu \leqslant 2$ such that $\langle y_n^2 \rangle = \infty$, the situation is different:

- For $0 < \mu < 1$ all moments of y_n diverge. Let us consider the characteristic function $g(p) = \langle \mathrm{e}^{-py} \rangle$. We can write $g(p) = 1 - \int_0^\infty \mathrm{d}y (1 - \mathrm{e}^{-py}) p(y)$ from which we see that

$g(p \to 0) \simeq 1 - Cp^{\mu}$ where C is some constant related to the prefactor of the power law tail of $p(y)$. The characteristic function for Y_N reads $G_N(p \to 0) \simeq \mathrm{e}^{-NCp^{\mu}}$. This shows that the related distribution $P_N(Y)$ presents a similar power law tail and involves the typical scale $Y_N \sim N^{1/\mu}$.

- For $\mu = 1$. A similar analysis gives $g(p \to 0) \simeq 1 - Cp \ln 1/p$ and therefore $Y_N \sim N \ln N$.
- For $1 < \mu < 2$ the first moment is finite $\langle Y_N \rangle = N\langle y \rangle$ however fluctuations are larger than in the normal case $\langle Y_N^2 \rangle_c \sim N^{2/\mu}$.
- For $\mu = 2$ fluctuations are $\langle Y_N^2 \rangle_c \sim N \ln N$.
- For $\mu > 2$, the central limit theorem applies.

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